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RANDOM FIELD IDENTIFICATION FROM A SAMPLE: 1 THE  
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FOR AUTOMATION RESEARCH M ROSENBLATT-ROTH NOV 85

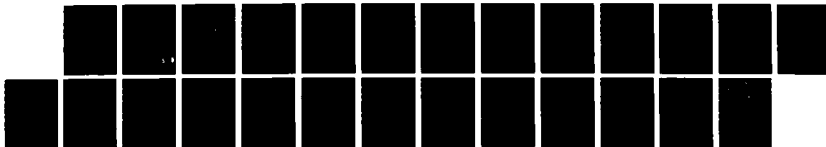
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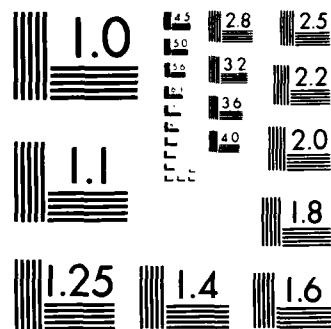
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19 ABSTRACT (Continue on reverse if necessary and identify by block number) <p>Given a random field belonging to some specific class, and given a data sample generated by the random field, we consider the problem of finding a field of the given class that approximates the field that generated the sample. This paper derives a solution to this problem for the simple case of a field consisting of independent random variables. Subsequent papers will treat other types of fields, e.g., having Markov dependencies. Numerical examples are given, showing that good approximations can be obtained based on relatively small sample sizes. In particular, this approach can be used to find random field models that generate given samples of image texture, and so can be applied to texture classification or segmentation.</p>																	
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RANDOM FIELD IDENTIFICATION FROM A SAMPLE:  
I. THE INDEPENDENT CASE

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ABSTRACT

Given a random field belonging to some specific class, and given a data sample generated by the random field, we consider the problem of finding a field of the given class that approximates the field that generated the sample. This paper derives a solution to this problem for the simple case of a field consisting of independent random variables. Subsequent papers will treat other types of fields, e.g., having Markov dependencies. Numerical examples are given, showing that good approximations can be obtained based on relatively small sample sizes. In particular, this approach can be used to find random field models that generate given samples of image texture, and so can be applied to texture classification or segmentation.

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## 1. Introduction

### 1.1. The purpose of this research

In what follows we will consider the following

**Problem:**

Given a sample, determine the random field that generated it.

At first glance, this problem seems to be without solution, because of the lack of sufficient data. In order to make the problem reasonable, it is necessary to assume that the field is not arbitrary but belongs to some specific class, e.g.,

- 1) is composed of independent random variables
- 2) is first order Markov (e.g., in two dimensions, it is a Kanak mesh [1] [2])
- 3) is  $n$ -th order Markov
- 4) is weakly (second order) stationary
- 5) is strongly stationary

and so on.

Making one of these assumptions means that in reality we are not considering the problem of finding the field that generated the given sample, but some other field that belongs to the given class and approximates the field that generated the given sample. In this paper we will not be interested in the problem of evaluating how good this approximation is, because this aspect is treated in the author's papers [8] [9].

In this series of papers we will study the problem in the case where the field is assumed to be strongly stationary, with some additional restrictions.

## **1.2. Direction of the research**

The first part of our research, presented in this paper, is concerned with the simplest case, where we have a stationary field made up of independent random variables; obviously, we may suppose that the field is one-dimensional.

The next stage of the research will consider the case of one-dimensional simple homogeneous Markov chains, followed by one-dimensional Markov homogeneous chains of higher order. Subsequent stages will study two- (or higher-) dimensional Markov random fields (Kanal meshes), simple or of higher orders.

## **1.3. Digitization**

In order to be able to deal with digitized data and at the same time to reduce the complexity of the problem, we will consider only random fields with a finite set of possible outcomes at each point. In order to extend these results to the continuous case, we would have to consider some process of approximation, such as that used by the author [3]-[7].

## **2. The direct theorem**

### **2.1. Generalities**

Let us consider a sequence of independent trials with possible outcomes  $A_i$  ( $1 \leq i \leq n$ ) and corresponding probabilities  $p_i > 0$  ( $1 \leq i \leq n$ ) adding up to 1. Each possible result of a series of  $s$  consecutive trials can be written as a

sequence

$$C_s = (A_{k_1}, A_{k_2}, \dots, A_{k_s}) \quad (2.1)$$

where each  $k_r$  ( $1 \leq r \leq s$ ) can take any value  $i$  ( $1 \leq i \leq n$ ). Because of stationarity, the probability of occurrence of the sequence  $C_s$  does not depend on the moment when the trials begin; taking into consideration the independence of the trials, this probability can be written as

$$P(C_s) = \prod_{r=1}^s P(A_{k_r}) \quad (2.2)$$

Let us denote by  $m_i$  ( $1 \leq i \leq n$ ) the number of times the outcome  $A_i$  appears in the sequence  $C_s$ , so that

$$\sum_{i=1}^n m_i = s. \quad (2.3)$$

The equality (2.2) can be written

$$P(C_s) = \prod_{i=1}^n p_i^{m_i} \quad (2.4)$$

In what follows we denote by

$$H = \sum_{i=1}^n p_i \log \frac{1}{p_i} \quad (2.5)$$

the entropy of the random field characterized by the probabilities  $p_i$  ( $1 \leq i \leq n$ ), and

$$\rho = \sum_{i=1}^n \log \frac{1}{p_i} \quad (2.6)$$

Obviously

$$0 < \rho < \infty \quad (2.7)$$

## 2.2. The theorem

Let us denote by  $\Gamma_s$  the class of all sequences  $C_s$ . For given  $\delta > 0$ ,  $s > 0$  we denote by  $\Gamma'_{\delta,s}$  the set of all sequences  $C_s \in \Gamma_s$  such that

$$|m_i - sp_i| < s\delta \quad (2.8)$$

for all  $i$  ( $1 \leq i \leq n$ ), and by  $\Gamma''_{\delta,s}$  its complement with respect to  $\Gamma_s$ .

**Definition.** Sequences  $C_s \in \Gamma'_{\delta,s}$  will be called  $(\delta,s)$ -standard sequences or simple standard sequences.

Let us consider the equation

$$\frac{1}{\sqrt{2\pi}} \int_0^u e^{-\frac{x^2}{2}} dx = \frac{1}{2} \left(1 - \frac{\epsilon}{n}\right) \quad (2.9)$$

and let us denote by  $u(\epsilon)$  its solution.

**Definition.** Given  $\epsilon > 0$ ,  $\delta > 0$ ,  $s > n$ , condition A holds if

$$4\delta^2 \epsilon s > n \quad (2.10)$$

and condition B holds if

$$4\delta^2 s > u^2(\epsilon) \quad (2.11)$$

Let us denote by  $N(\cdot)$  the cardinality of a set.



**Theorem 1.**

Let us suppose that at least one of the conditions A, B holds. Then

(a) If  $C_s$  is a  $(\delta, s)$ -standard sequence, it follows that

$$\left| \frac{1}{s} \log \frac{1}{P(C_s)} - H \right| < \delta \rho \quad (2.12)$$

$$(b) P(\Gamma'_{\delta, s}) \geq 1 - \epsilon \quad (2.13)$$

$$(c) \lim_{\substack{s \rightarrow \infty \\ \delta \rightarrow 0}} \frac{1}{s} \log N(\Gamma'_{\delta, s}) = H \quad (2.14)$$

**Remark 1.**

The relation (2.12) is equivalent to

$$2^{-s(H + \delta \rho)} < P(C_s) < 2^{-s(H - \delta \rho)} \quad (2.15)$$

i.e. to

$$P(C_s) = 2^{-sH + s\delta\rho\theta}, \quad |\theta| < 1 \quad (2.16)$$

**Remark 2.**

The relation (2.13) is equivalent to

$$P(\Gamma''_{\delta, s}) < \epsilon \quad (2.17)$$

**Remark 3.**

From (2.14) it follows that

$$\lim_{s \rightarrow \infty} \frac{N(\Gamma'_{\delta, s})}{N(\Gamma_s)} = 0 \quad , \quad \lim_{s \rightarrow \infty} \frac{N(\Gamma''_{\delta, s})}{N(\Gamma_s)} = 1 \quad (2.18)$$

Indeed, from (2.14) we obtain the relation

$$\log N(\Gamma'_{\delta,s}) = s \cdot (H + o(1)) \quad (2.19)$$

i.e.,

$$N(\Gamma'_{\delta,s}) = 2^{s(H + o(1))} \quad (2.20)$$

Taking into consideration that

$$N(\Gamma_s) = n^s = 2^{s \log n} \quad (2.21)$$

and because

$$H < \log n, \quad (2.22)$$

it follows that

$$\frac{N(\Gamma'_{\delta,s})}{N(\Gamma_s)} = 2^{-s(\log n - H + o(1))} = o(1) \quad (2.23)$$

which is equivalent to the first equality in (2.18), and

$$\frac{N(\Gamma''_{\delta,s})}{N(\Gamma_s)} = \frac{N(\Gamma_s) - N(\Gamma'_{\delta,s})}{N(\Gamma_s)} = 1 - \frac{N(\Gamma'_{\delta,s})}{N(\Gamma_s)} = 1 + o(1) \quad (2.24)$$

which is equivalent to the second equality in (2.18).

**Remark 4.**

Our Theorem 1 is closely related to some results which go back to Shannon [10] and received a mathematically acceptable form from Khinchine [2].

Our Theorem 1(a), (b) refers to independent random variables, while that in [2] refers to ergodic simple Markov chains, but our result is not a particular case of that in [2]. Indeed, the results in [2] are existence theorems, considering that  $\delta$ ,  $\epsilon$  can be taken as small and  $s$  as large as desired, while our results give effective relations between  $\delta$ ,  $\epsilon$ ,  $s$  in order that the results hold.

Our Theorem 1(c) refers to the set  $\Gamma'_{\delta,s}$  of all standard sequences  $C_s$ , while the result in ([2], Th. 3) refers to another set of sequences  $C_s$ ; our result contains a limit for  $\delta \rightarrow 0$ ,  $s \rightarrow \infty$ , while the result in ([2], Th. 3) contains a limit for  $s \rightarrow \infty$ .

### 2.3. Proof

(a) Let us consider a sequence  $C_s \in \Gamma'_{\delta,s}$ . From (2.8) it follows that

$$m_i = sp_i + s\delta\theta_i \quad |\theta_i| < 1 \quad (1 \leq i \leq n) \quad (2.25)$$

From (2.4) there follows the relation

$$\log P(C_s) = \sum_{i=1}^n m_i \log p_i \quad (2.26)$$

and taking into consideration (2.25), there follows the equality

$$\begin{aligned} \log P(C_s) &= \sum_{i=1}^n (sp_i + s\delta\theta_i) \log p_i \\ &= s \sum_{i=1}^n p_i \log p_i + s\delta \cdot \sum_{i=1}^n \theta_i \log p_i \end{aligned} \quad (2.27)$$

which can also be written as

$$\log \frac{1}{P(C_s)} = sH + s\delta \cdot \sum_{i=1}^n \theta_i \log \frac{1}{p_i} \quad (2.28)$$

From (2.28) we obtain the result (a):

$$\left| \frac{1}{s} \log \frac{1}{P(C_s)} - H \right| < \delta \cdot \sum_{i=1}^n |\theta_i| \log \frac{1}{p_i} \leq \delta \cdot \sum_{i=1}^n \log \frac{1}{p_i} = \delta\rho \quad (2.29)$$

(b) Instead of proving inequality (2.13) we will prove (2.17). In order that a sequence  $C_s \in \Gamma_s$  belong to  $\Gamma''_{\delta,s}$ , it is necessary that for at least some value of  $i$  ( $1 \leq i \leq n$ ) the inequality (2.8) does not hold, i.e.,

$$\Gamma''_{\delta,s} = \bigcup_{i=1}^n \left\{ |m_i - sp_i| > s\delta \right\} \quad (2.30)$$

so that

$$P(\Gamma''_{\delta,s}) = P\left\{ \bigcup_{i=1}^n \left\{ |m_i - sp_i| > s\delta \right\} \right\} \leq \sum_{i=1}^n P\left\{ |m_i - sp_i| > s\delta \right\} \quad (2.31)$$

(b1) Let us assume that condition A holds. It is known from the elements of the Theory of probability that

$$P\left\{ |m_i - sp_i| > s\delta \right\} \leq \frac{p_i(1-p_i)}{s\delta^2} \quad (2.32)$$

But for  $0 \leq x \leq 1$ , we have the inequalities

$$0 \leq x(1-x) \leq \frac{1}{4} \quad (2.33)$$

where the maximum value is reached for  $x = \frac{1}{2}$ , so that from (2.32) it follows that

$$P\left\{ |m_i - sp_i| > s\delta \right\} \leq \frac{1}{4s\delta^2} \quad (1 \leq i \leq n) \quad (2.34)$$

Consequently, from (2.31) there follows the inequality

$$P(\Gamma''_{\delta,s}) \leq \frac{n}{4s\delta^2} \quad (2.35)$$

and because of (2.10), it follows that (2.17) holds.

(b2) Let us assume that condition B holds. From the Central limit theorem in the Moivre-Laplace form, it is known that

$$P\left\{\left|\frac{m_i}{s} - p_i\right| \leq \delta\right\} = P\left\{\left|\frac{m_i - sp_i}{\sqrt{sp_i(1-p_i)}}\right| \leq \delta\sqrt{\frac{s}{p_i(1-p_i)}}\right\} \\ \sim \frac{2}{\sqrt{2\pi}} \int_0^{\delta\sqrt{\frac{s}{p_i(1-p_i)}}} e^{-\frac{x^2}{2}} dx \quad (1 \leq i \leq n) \quad (2.36)$$

so that

$$P\left\{|m_i - sp_i| > \delta s\right\} \sim 1 - \frac{2}{\sqrt{2\pi}} \int_0^{\delta\sqrt{\frac{s}{p_i(1-p_i)}}} e^{-\frac{x^2}{2}} dx, \quad (1 \leq i \leq n) \quad (2.37)$$

In order to obtain the relation (2.17) it is sufficient to take

$$1 - \frac{2}{\sqrt{2\pi}} \int_0^{\delta\sqrt{\frac{s}{p_i(1-p_i)}}} e^{-\frac{x^2}{2}} dx < \frac{\epsilon}{n} \quad (1 \leq i \leq n) \quad (2.38)$$

i.e.,

$$\frac{1}{\sqrt{2\pi}} \int_0^{\delta\sqrt{\frac{s}{p_i(1-p_i)}}} e^{-\frac{x^2}{2}} dx > \frac{1}{2} \cdot \left[1 - \frac{\epsilon}{n}\right] \quad (1 \leq i \leq n) \quad (2.39)$$

which is equivalent to the inequality

$$\delta\sqrt{\frac{s}{p_i(1-p_i)}} > u(\epsilon) \quad (1 \leq i \leq n) \quad (2.40)$$

Because of (2.33), we have the inequality

$$\delta\sqrt{\frac{s}{p_i(1-p_i)}} \geq 2\delta\sqrt{s} \quad (1 \leq i \leq n) \quad (2.41)$$

so that in order to satisfy (2.40) it is sufficient to take in consideration Condition B (2.11), i.e.

$$2\delta\sqrt{s} > u(\epsilon) \quad (2.42)$$

(c) If  $C_s \in \Gamma'_{\delta,s}$ , then (2.15) holds, so that

$$N(\Gamma'_{\delta,s}) 2^{-s(H+\delta\rho)} < \sum P(C_s) = P(\Gamma'_{\delta,s}) < 1 \quad (2.43)$$

where the summation is for all  $C_s \in \Gamma'_{\delta,s}$ . From (2.43) there follows the relation

$$\frac{1}{s} \cdot \log N(\Gamma'_{\delta,s}) < H + \delta\rho \quad (2.44)$$

In a similar way, from (2.13), (2.15) there follow the relations

$$1 - \epsilon < P(\Gamma'_{\delta,s}) = \sum P(C_s) < N(\Gamma'_{\delta,s}) 2^{-s(H-\delta\rho)} \quad (2.45)$$

where the summation is also for all  $C_s \in \Gamma'_{\delta,s}$ . From (2.45) we obtain the relation

$$H - \delta\rho < \frac{1}{s} \cdot \log N(\Gamma'_{\delta,s}) + \frac{1}{s} \log \frac{1}{1 - \epsilon} \quad (2.46)$$

From (2.44), (2.46) it follows that

$$H - \delta\rho - \frac{1}{s} \cdot \log \frac{1}{1 - \epsilon} < \frac{1}{s} \log N(\Gamma'_{\delta,s}) < H + \delta\rho \quad (2.47)$$

For  $\epsilon$  given, arbitrary,  $\delta$  as small as we want, and  $s$  as large as we want, because of (2.7) it follows that (2.14) holds.

### 3. The inverse theorem

#### 3.1. Generalities

Let  $\delta > 0$ ,  $\epsilon > 0$ ,  $s > 1$ , and let  $C_s^0$  be an arbitrary specific sequence, belonging to  $\Gamma_s$ . Let us assume that one of the conditions A or B holds.

In what follows we assume that  $C_s^0$  is generated by a sequence of independent trials, with possible outcomes  $A_i$  ( $1 \leq i \leq n$ ) with unknown probabilities  $p_i$  ( $1 \leq i \leq n$ ), and we will try to determine some intervals in which these probabilities can take values. Let us denote

$$m_i^0 = m_i(C_s^0) \quad (1 \leq i \leq n) \quad (3.1)$$

and by  $W\{S\}$  the confidence of statement  $S$ .

#### 3.2. The theorem

Because we have proved that

$$P(\Gamma'_{\delta,s}) > 1 - \epsilon, \quad P(\Gamma''_{\delta,s}) < \epsilon \quad (3.2)$$

it follows that with confidence larger than  $1 - \epsilon$ ,  $C_s^0 \in \Gamma'_{\delta,s}$ , i.e.,

$$W\left\{ |m_i^0 - sp_i| < \delta s, \quad (1 \leq i \leq n) \right\} > 1 - \epsilon \quad (3.3)$$

i.e.,

$$W\left\{ \frac{m_i^0}{s} - \delta < p_i < \frac{m_i^0}{s} + \delta, \quad (1 \leq i \leq n) \right\} > 1 - \epsilon \quad (3.4)$$

Let  $L_n$  be the Banach space of all vectors

$$q = (q_1, \dots, q_n) \quad (3.5)$$

with  $q_i$  real numbers of any sign, with norm

$$\|q\| = \sup \left\{ |q_i|; 1 \leq i \leq n \right\} \quad (3.6)$$

Let  $\Pi_n$  be the totality of probability measures

$$p = (p_1, \dots, p_n) \quad (3.7)$$

with  $p_i > 0$  ( $1 \leq i \leq n$ ), and

$$\sum_{i=1}^n p_i = 1 \quad (3.8)$$

This is a metric space with distance

$$\|p - p'\| = \sup \left\{ |p_i - p'_i|; 1 \leq i \leq n \right\} \quad (3.9)$$

where  $p, p' \in \Pi_n$ ,  $p - p' \in L_n$ . If  $p, p' \in \Pi_n$  are two different solutions, satisfying the inequalities in (3.4), it follows that

$$|p_i - p'_i| < 2\delta \quad (1 \leq i \leq n) \quad (3.10)$$

so that from (3.9) it follows that

$$\|p - p'\| < 2\delta \quad (3.11)$$

We have thus proved

## **Theorem 2.**

Let us assume that



- (1)  $\epsilon, \delta, s$  satisfy one of the conditions A, B;
- (2) the arbitrary sequence  $C_s^0 \in \Gamma_s$  is generated by an independent identically distributed sequence of trials, with unknown probabilities  $p_i$  ( $1 \leq i \leq n$ ).

Then

- (a) The relation (3.4) holds.
- (b) If  $p, p'$  are two different solutions, their distance in  $\Pi_n$  is less than  $2\delta$ .

**Remark 4.**

Let  $L'_n$  be the Banach space of all vectors (3.5) with norm the total variation

$$|||q||| = \sum_{i=1}^n |q_i| \quad (3.12)$$

Then  $\Pi_n$  is a metric space with distance

$$|||p - p'||| = \sum_{i=1}^n |p_i - p'_i| \quad (3.13)$$

where  $p, p' \in \Pi_n$ .

If  $p, p' \in \Pi_n$  are two different solutions, satisfying (3.4), it follows from (3.13) that

$$|||p - p'||| < 2n\delta \quad (3.14)$$

It is easy to see that

$$||p - p'|| \leq |||p - p'||| \leq n||p - p'|| \quad (3.15)$$

We remark also that if  $L''_n$  is the Euclidean space of all vectors (3.5) with norm

$$((q)) = \left( \sum_{i=1}^n q_i^2 \right)^{\frac{1}{2}} \quad (3.16)$$

then  $\Pi_n$  is a Euclidean space with distance

$$((p - p')) = \left( \sum_{i=1}^n |p_i - p'_i|^2 \right)^{\frac{1}{2}} \quad (3.17)$$

It is easy to see that

$$\|p - p'\| \leq ((p - p')) \leq \sqrt{n} \|p - p'\| \quad (3.18)$$

## 4. Examples

### 4.1. Examples under Condition A

#### Example 1.

Let  $C_s^0$  be a sequence with  $n = 2$ ,  $s = 10^4$ ,  $\epsilon = 2^{-3} = 0.125$ ,  $\delta > 0.02$ , so that condition A holds. Let  $m_1^0 = 3 \times 10^3$ ,  $m_2^0 = 7 \times 10^3$ .

From (3.4) it follows that

$$W \left\{ 0.28 < p_1 < 0.32; \quad 0.68 < p_2 < 0.72 \right\} > 0.875 \quad (4.1)$$

and from (3.11) we obtain

$$\|p - p'\| < 0.04 \quad (4.2)$$

#### Example 2.

Let  $C_s^0$  be a sequence with  $n = 2$ ,  $s = 10^6$ ,  $\epsilon = 2^{-3} = 0.125$ ,  $\delta > 0.002$ , so that condition A holds. Let  $m_1^0 = 3 \times 10^5$ ,  $m_2^0 = 7 \times 10^5$ .

From (3.4) it follows that

$$W\left\{ 0.298 < p_1 < 0.302; \quad 0.698 < p_2 < 0.702 \right\} > 0.875 \quad (4.3)$$

and from (3.11) we obtain

$$\|p - p'\| < 0.004 \quad (4.4)$$

## 4.2. Examples under Condition B

### Example 3.

Let  $C_s^0$  be a sequence with  $n = 2$ ,  $\epsilon = 2^{-3} = 0.125$ ,  $s = 10^4$ ,  $\delta > 0.009$ ,

$m_1^0 = 3 \times 10^3$ ,  $m_2^0 = 7 \times 10^3$ , so that

$$\frac{1}{2} \left( 1 - \frac{\epsilon}{2} \right) = 0.46875 \quad (4.5)$$

and relation (2.39) takes the form

$$\frac{1}{\sqrt{2\pi}} \int_0^{u(\epsilon)} e^{-\frac{x^2}{2}} dx > 0.46875 \quad (4.6)$$

which holds for

$$u(\epsilon) > 1.8 \quad (4.7)$$

Considering Condition B in form (1.42) it is easy to see that it holds. From

(3.4) it follows that

$$W\left\{ 0.291 < p_1 < 0.309; \quad 0.691 < p_2 < 0.709 \right\} > 0.875 \quad (4.8)$$

and from (3.11) it follows that

$$\|p - p'\| < 0.018 \quad (4.9)$$

**Example 4.**

Let  $C_s^0$  be a sequence with  $n = 2$ ,  $\epsilon = 2^{-3} = 0.125$ ,  $s = 10^6$ ,  $\delta > 0.0009$ ,  $m_1^0 = 3 \times 10^5$ ,  $m_2^0 = 7 \times 10^5$ ; in this case, relations (4.5)–(4.8) hold, so that Condition B holds. From (3.4) it follows that

$$W\left\{ 0.2991 < p_1 < 0.3009 ; 0.6991 < p_2 < 0.7009 \right\} > 0.875 \quad (4.10)$$

and from (3.11) it follows that

$$\|p - p'\| < 0.0018 \quad (4.11)$$

### 4.3. Examples involving images that satisfy Condition A or B

**Example 5.**

Let us consider a digital television picture, i.e., an array of  $500^2$  points, where each point can have 256 levels of gray.

Here  $n = 256$ ,  $s = 500^2 = 250,000$ ; let  $\epsilon = \frac{1}{256} = 0.00390625$ .

Taking these values, if we want Condition A satisfied it is sufficient that

$$4\delta^2 \times 250,000 \times \frac{1}{256} > 256 \quad (4.12)$$

or

$$10^6 \delta^2 > 256^2, \quad (4.13)$$

i.e.,

$$\delta > 0.256 \quad (4.14)$$

Consequently

$$W\left\{\left|\frac{m_i^0}{s} - p_i\right| < 0.256 ; (1 \leq i \leq 256)\right\} > 0.9960937 \quad (4.15)$$

with

$$\|p - p'\| = \max\{|p_i - p'_i|, 1 \leq i \leq 256\} < 0.512 \quad (4.16)$$

Example 6.

With the same basic data as in Example 5, we take  $n = 256$ ,  $s = 500^2$ ,

$\epsilon = \frac{1}{256} = 0.00390625$ , and we consider that Condition B holds, i.e.,

$$2\delta\sqrt{s} > u(\epsilon) \quad (4.17)$$

Here

$$\begin{aligned} \frac{1}{2} \left(1 - \frac{\epsilon}{n}\right) &= \frac{1}{2} \left(1 - \frac{1}{256^2}\right) = \frac{1}{2} \left(1 - \frac{1}{65,536}\right) \\ &\sim \frac{1}{2} \left(1 - \frac{1}{60,000}\right) \sim \frac{1}{2} (1 - 0.16667) = \frac{1}{2} \times 0.83334 \\ &= 0.41667 \end{aligned} \quad (4.18)$$

so that from tables it follows that

$$u(\epsilon) \sim 1.30 \quad (4.19)$$

Thus

$$2\delta \times 500 > 1.30 \quad (4.20)$$

i.e.,

$$\delta > 0.0013 \quad (4.21)$$

So

$$W\left\{\left|\frac{m_i^0}{s} - p_i\right| < 0.0013 ; \quad (1 \leq i \leq 256)\right\} > 0.9960937 \quad (4.22)$$

and

$$\|p - p'\| < 0.0026 \quad (4.23)$$

Example 7.

Let us take  $n = 256$ ,  $s = 500^2$ ,  $\epsilon = \frac{1}{16} = 0.0625$ , and let us assume that

Condition A holds. Then

$$4\delta^2 \times 250,000 \times \frac{1}{16} > 256 \quad (4.24)$$

i.e.,

$$10^6 \delta^2 > 2^{12} \quad (4.25)$$

or

$$\delta > 0.064 \quad (4.26)$$

so that

$$W\left\{\left|\frac{m_i^0}{s} - p_i\right| < 0.064 ; \quad 1 \leq i \leq 256\right\} > 0.9375 \quad (4.27)$$

$$\|p - p'\| < 0.128 \quad (4.28)$$

Example 8.

Let  $n = 256$ ,  $s = 250,000$ ,  $\epsilon = \frac{1}{16} = 0.0625$  and let us assume that Condi-

tion B holds. Then

$$\begin{aligned} \frac{1}{2} \left( 1 - \frac{\epsilon}{n} \right) &= \frac{1}{2} \left( 1 - \frac{1}{16} \cdot \frac{1}{256} \right) = \frac{1}{2} \left( 1 - \frac{1}{4096} \right) \sim \frac{1}{2} \left( 1 - \frac{1}{4000} \right) \\ &= \frac{1}{2} (1 - 0.00025) = \frac{1}{2} \times 0.99975 = 0.49987 \end{aligned} \quad (4.29)$$

so that

$$u(\epsilon) \sim 3.8 \quad (4.30)$$

i.e.,

$$2\delta \times 500 > 3.8 \quad (4.31)$$

or

$$\delta > 0.0038 \quad (4.32)$$

Thus

$$W \left\{ \left| \frac{m_i^0}{s} - p_i \right| < 0.0038 ; 1 \leq i \leq 256 \right\} > 0.9375 \quad (4.33)$$

$$\|p - p'\| < 0.0076 \quad (4.34)$$

#### Example 9.

Let us assume that we have a 30-minute sequence of TV pictures. If we have 32 pictures in each second, we have a total of

$$32 \times 60 \times 30 = 2^4 \times 60^2 \quad (4.35)$$

pictures, succeeding each other in time. Assuming independence between the pictures, we have  $n = 256$ ,  $s = 500^2 \times 2^4 \times 60^2$ , and let  $\epsilon = \frac{1}{256} = 0.00390625$ . Assuming that Condition A holds, the value of  $\delta$  is given by

$$4\delta^2 (250,000) \times 2^4 \times 60^2 \times \frac{1}{256} > 256 \quad (4.36)$$

or

$$10^6 \delta^2 \times 2^4 \times 60^2 > 256^2 \quad (4.37)$$

i.e.,

$$10^3 \delta \times 2^2 \times 60 > 256 \quad (4.38)$$

Then

$$\delta > \frac{256}{10^2 \times 240} > 0.001 \quad (4.39)$$

Consequently

$$W \left\{ \left| \frac{m_i^0}{s} - p_i \right| < 0.001 ; (1 \leq i \leq 256) \right\} > 0.9960937 \quad (4.40)$$

and

$$\|p - p'\| < 0.002 \quad (4.41)$$

#### Example 10.

Let us consider the same problem as in Example 9, with the supposition that Condition B holds.

In this case

$$2\delta (500 \times 2^2 \times 60) > 1.30 \quad (4.42)$$

i.e.,

$$\delta > \frac{13}{2,400,000} \sim 0.0000054 \quad (4.43)$$

so that

$$W \left\{ \left| \frac{m_i^0}{s} - p_i \right| < 0.0000054 ; (1 \leq i \leq 256) \right\} > 0.9960937 \quad (4.44)$$

and

$$\|p - p'\| < 0.0000108 \quad (4.45)$$



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